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Generalize of some Inequality related to the gamma functions

Sudprathai Bupasiri *

Faculty of Education, Sakon Nakhon Rajabhat University, Sakon Nakhon, 47000 Thailand

***Corresponding Author**: sudprathai@gmail.com

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Abstract

We present some generalized inequalities involving the ratios of analogues of the gamma and some monotonic function

$$\Lambda(t)=\frac{t^{(a+b)\beta}e^{-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{p^{-(ab)\beta t}\Gamma_p(\alpha+\beta t)^{(ab)}},\ t\in(0,\infty),\ k>0,\ p\in N$$

and

$$\Upsilon(t)=\frac{t^{(a+b)\beta}e^{-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{(1-q)^{(ab)\beta t}\Gamma_q(\alpha+\beta t)^{(ab)}},\ t\in(0,\infty),\ k>0,\ p\in(0,1).$$

Keywords: Digamma function; Polygamma function; Generalized inequalities

1. Introduction

It is well-known that Euler gamma function $\Gamma(x)$ is defined for $x>0$ by

$$\Gamma(x)=\int_0^\infty e^{-t}t^{x-1}dt. \qquad (1)$$

The logarithmic derivative of $\Gamma(x)$ denote by $\psi(x)=\frac{\Gamma'(x)}{\Gamma(x)}$ is call the psi or digamma function $\psi^{(k)}(x)$ for $k\in N$ are call the polygamma function.

The p-digamma function $\psi_p(t)$, q-digamma function $\psi_q(t)$ and k-digamma function $\psi_k(t)$ are respectively defined as follows.

$$\psi_p(t) = \frac{d}{dt}\ln\left(\Gamma_p(t)\right) = \frac{\Gamma'_p(t)}{\Gamma_p(t)}, \quad t > 0 \tag{2}$$

where $\Gamma_p(t)$ is the p-analogue of the gamma function defined by [1]

$$\Gamma_p(t) = \frac{p!\,p^t}{t(t+1)\cdots(t+p)} = \frac{p^t}{t\left(1+\frac{t}{1}\right)\cdots\left(1+\frac{t}{p}\right)}, \quad p \in N,\ t > 0 \tag{3}$$

where p is positive integer, and

$$\Gamma(x) = \lim_{p\to\infty} \Gamma_p(x). \tag{4}$$

The p-analogue of the psi function, as the logarithmic derivative of the Γ_p function, is

$$\psi_p = \frac{d}{dx}\ln\Gamma_p(x) = \frac{\Gamma'_p(x)}{\Gamma_p(x)}. \tag{5}$$

The following representations are valid :

$$\Gamma_p(x) = \int_0^\infty \left(1-\frac{t}{p}\right)^p t^{x-1}\,dt \tag{6}$$

$$\psi_p(x) = \ln p - \int_0^\infty \frac{e^{-xt}\left(1-e^{-(p+1)t}\right)}{1-e^{-t}}\,dt, \tag{7}$$

and

$$\psi_p^{(m)}(x) = (-1)^{m+1}\int_0^\infty \frac{t^m e^{-xt}}{1-e^{-t}}\left(1-e^{-pt}\right)dt, \tag{8}$$

$$\psi_q(t) = \frac{d}{dt}\ln\left(\Gamma_q(t)\right) = \frac{\Gamma'_q(t)}{\Gamma_q(t)}, \quad t > 0 \tag{9}$$

where $\Gamma_q(t)$ is the q-analogue of the gamma function defined by [2]

$$\Gamma_q(t) = (1-q)^{1-t}\prod_{n=1}^{\infty}\frac{1-q^n}{1-q^{t+n}}, \quad q \in (0,1),\ t > 0. \tag{10}$$

The q-gamma function has the following integral representation

$$\Gamma_q(t) = \int_0^\infty x^{t-1}E_q^{-qx}\,d_q x, \tag{11}$$

where

$$E_q^x = \sum_{j=0}^{\infty} q^{\frac{j(j-1)}{2}}\frac{x^j}{|j|!} = \left(1+(1-q)x\right)_q^\infty \text{ and } \psi_k(t) = \frac{d}{dt}\ln\left(\Gamma_k(t)\right) = \frac{\Gamma'_k(t)}{\Gamma_k(t)}, \quad t > 0$$

where $\Gamma_k(t)$ is the k-analogue of the gamma function defined by [3, 4]

$$\Gamma_k(t) = \int e^{-\frac{x^k}{k}} x^{t-1} dx, \; k>0, \; p \in N. \tag{12}$$

In a recent paper [5], Nantomah and Iddrisu proved that the following double inequalities hold:

$$0 < \frac{\Gamma_k(\alpha+t)}{\Gamma_p(\alpha+t)} < \frac{e^{\left(e^{t-1}\right)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+1)}{tp^{t-1}\Gamma_p(\alpha+1)}, \; k>0, \; p \in N \tag{13}$$

$$0 < \frac{\Gamma_k(\alpha+t)}{\Gamma_q(\alpha+t)} < \frac{e^{\left(e^{-t}\right)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+1)}{t(1-q)^{1-t}\Gamma_q(\alpha+1)}, \; k>0, \; q \in (0,1) \tag{14}$$

for $t \in (0,1)$ and for a positive real number α. Our objective in this paper is to establish some generalizations of the inequalities (13) and (14).

2. Preliminaries

The following auxiliary results are crucial to the main results of the paper.

Lemma 2.1. The functions $\psi_k(t)$, $\psi_q(t)$ and $\psi_p(t)$ as defined above have the following series representations

$$\psi_p(t) = \ln p - \sum_{n=0}^{p} \frac{1}{n+t}, \quad p \in N, \; t>0 \tag{15}$$

$$\psi_q(t) = -\ln(1-q) + \ln p \sum_{n=0}^{p} \frac{q^{t+n}}{1-q^{t+n}}, \quad q \in (0,1), \; t>0 \tag{16}$$

$$\psi_k(t) = \frac{\ln k - \gamma}{k} - \frac{1}{t} + \sum_{n=1}^{\infty} \frac{q^t}{nk(nk+1)}, \quad k>0, \; t>0 \tag{17}$$

where γ is the Euler-Mascheroni's constant.

Proof. See [4 – 6] and the references therein.

Lemma 2.2. Let $a>0$, $b>0$ and $t>0$. Then,

$$-(a+b)\frac{\ln k-\gamma}{k} + (ab)\ln p + \frac{a+b}{t} + (a+b)\psi_k(t) - (ab)\psi_p(t) > 0.$$

Proof. Using the series representations in equations (15) and (17) we have,

$$-(a+b)\frac{\ln k-\gamma}{k} + (ab)\ln p + \frac{a+b}{t} + (a+b)\psi_k(t) - (ab)\psi_p(t) > 0$$

$$= (a+b)\sum_{n=1}^{\infty}\frac{q^{t}}{nk(nk+1)} + (ab)\sum_{n=0}^{p}\frac{1}{n+t} > 0.$$

Lemma 2.3. Let $a>0$, $b>0$ and $\alpha+\beta t>0$. Then,

$$-(a+b)\frac{\ln k-\gamma}{k} + (ab)\ln p + \frac{a+b}{t} + (a+b)\psi_k(\alpha+\beta t) - (ab)\psi_p(\alpha+\beta t) > 0.$$

Proof. This follows directly from Lemma 2.2.

Lemma 2.4. Let $a>0$, $b>0$ and $t>0$. Then,

$$-(a+b)\frac{\ln k-\gamma}{k} + (ab)\ln(1-q) + \frac{a+b}{t} + (a+b)\psi_k(t) - (ab)\psi_q(t) > 0.$$

Proof. Using the series representations in equations (16) and (17) we have,

$$-(a+b)\frac{\ln k-\gamma}{k} + (ab)\ln(1-q) + \frac{a+b}{t} + (a+b)\psi_k(t) - (ab)\psi_q(t)$$

$$= (a+b)\sum_{n=1}^{\infty}\frac{q^{t}}{nk(nk+1)} + (ab)\ln q\sum_{n=0}^{p}\frac{q^{t+n}}{1-q^{t+n}} > 0.$$

Lemma 2.5. Let $a>0$, $b>0$ and $\alpha+\beta t>0$. Then,

$$-(a+b)\frac{\ln k-\gamma}{k} + (ab)\ln(1-q) + \frac{a+b}{t} + (a+b)\psi_k(\alpha+\beta t) - (ab)\psi_p(\alpha+\beta t) > 0.$$

Proof. This follows directly from Lemma 2.4.

3. Main results

Theorem 3.1. Define a function Λ by

$$\Lambda(t) = \frac{t^{(a+b)\beta} e^{-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{p^{-(ab)\beta t}\Gamma_p(\alpha+\beta t)^{(ab)}},\ t\in(0,\infty),\ k>0,\ p\in N$$

where a,b,α,β are positive real numbers. Then Λ is increasing on $t\in(0,\infty)$ and the inequality

$$0 < \frac{\Gamma_k(\alpha+\beta t)^{(a+b)}}{\Gamma_p(\alpha+\beta t)^{(ab)}} < \frac{e^{(a+b)\beta(t-1)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{t^{(a+b)\beta}p^{(ab)\beta(t-1)}\Gamma_p(\alpha+\beta t)^{(ab)}}.$$

Proof. Let $g(t)=\ln\Lambda(t)$ for every $t\in(0,\infty)$. Then

$$\Lambda(t)=\ln\frac{t^{(a+b)\beta}e^{-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{p^{-(ab)\beta t}\Gamma_p(\alpha+\beta t)^{(ab)}}$$

$$=-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)$$
$$+(ab)\beta t\ln p+(a+b)\beta\ln t+(a+b)\ln\Gamma_k(\alpha+\beta t)-(ab)\ln\Gamma_p(\alpha+\beta t).$$

Then,

$$g'(t)=-(a+b)\beta\left(\frac{\ln k-\gamma}{k}\right)$$
$$+(ab)\beta\ln p+\frac{(a+b)\beta}{t}+(a+b)\beta\Psi_k(\alpha+\beta t)-(ab)\beta\Psi_p(\alpha+\beta t)$$
$$=\beta\left[-(a+b)\left(\frac{\ln k-\gamma}{k}\right)\right.$$
$$\left.+(ab)\ln p+\frac{(a+b)}{t}+(a+b)\Psi_k(\alpha+\beta t)-(ab)\Psi_p(\alpha+\beta t)\right]>0$$

as a result of Lemma 2.3. This proves that g is increasing on $t\in(0,\infty)$. Hence Λ is increasing on $t\in(0,\infty)$. Thus, for every $t\in(0,1)$ we have $\Lambda(0)<\Lambda(t)<\Lambda(1)$, yielding

$$0<\frac{\Gamma_k(\alpha+\beta t)^{(a+b)}}{\Gamma_p(\alpha+\beta t)^{(ab)}}<\frac{e^{(a+b)\beta(t-1)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{t^{(a+b)\beta}p^{(ab)\beta(t-1)}\Gamma_p(\alpha+\beta t)^{(ab)}}.$$

The proof is completed.

Corollary 3.1. If $t\in[1,\infty)$, then the following inequality holds

$$\frac{e^{(a+b)\beta(t-1)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{t^{(a+b)\beta}p^{(ab)\beta(t-1)}\Gamma_p(\alpha+\beta t)^{(ab)}}\leq\frac{\Gamma_k(\alpha+\beta t)^{(a+b)}}{\Gamma_p(\alpha+\beta t)^{(ab)}}.$$

Proof. If $t\in[1,\infty)$, then we have $\Lambda(1)\leq\Lambda(t)$ yielding the result.

Theorem 3.2. Define a function Υ by

$$\Upsilon(t)=\frac{t^{(a+b)\beta}e^{-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{(1-q)^{(ab)\beta t}\Gamma_q(\alpha+\beta t)^{(ab)}},\ t\in(0,\infty),\ k>0,\ q\in(0,1)$$

where a,b,α,β are positive real numbers. Then Υ is increasing on $t\in(0,\infty)$ and the inequality

$$0 < \frac{\Gamma_k(\alpha+\beta t)^{(a+b)}}{\Gamma_q(\alpha+\beta t)^{(ab)}} < \frac{e^{(a+b)\beta(t-1)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{t^{(a+b)\beta}(1-q)^{(ab)\beta(1-t)}\Gamma_q(\alpha+\beta t)^{(ab)}}.$$

Proof. Let $h(t)=\ln \Upsilon(t)$ for every $t\in(0,\infty)$. Then

$$h(t)=\ln\frac{t^{(a+b)\beta}e^{-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{(1-q)^{(ab)\beta t}\Gamma_q(\alpha+\beta t)^{(ab)}}$$

$$=-(a+b)\beta t\left(\frac{\ln k-\gamma}{k}\right)$$

$$-(ab)\beta t\ln(1-q)+(a+b)\beta\ln t+(a+b)\ln\Gamma_k(\alpha+\beta t)-(ab)\ln\Gamma_q(\alpha+\beta t).$$

Then,

$$h'(t)=-(a+b)\beta\left(\frac{\ln k-\gamma}{k}\right)$$

$$-(ab)\beta\ln(1-q)+\frac{(a+b)\beta}{t}+(a+b)\beta\Psi_k(\alpha+\beta t)-(ab)\beta\Psi_q(\alpha+\beta t)$$

$$=\beta\left[-(a+b)\left(\frac{\ln k-\gamma}{k}\right)\right.$$

$$\left.-(ab)\ln(1-q)+\frac{(a+b)}{t}+(a+b)\Psi_k(\alpha+\beta t)-(ab)\Psi_q(\alpha+\beta t)\right]>0$$

as a result of Lemma 2.5. This proves that h is increasing on $t\in(0,\infty)$. Hence Υ is increasing on $t\in(0,\infty)$. Thus, for every $t\in(0,1)$ we have $\Upsilon(0)<\Upsilon(t)<\Upsilon(1)$, yielding

$$0 < \frac{\Gamma_k(\alpha+\beta t)^{(a+b)}}{\Gamma_q(\alpha+\beta t)^{(ab)}} < \frac{e^{(a+b)\beta(t-1)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{t^{(a+b)\beta}(1-q)^{(ab)\beta(t-1)}\Gamma_q(\alpha+\beta t)^{(ab)}}.$$

The proof is completed.

Corollary 3.2. If $t\in[1,\infty)$, then the following inequality holds

$$\frac{e^{(a+b)\beta(t-1)\left(\frac{\ln k-\gamma}{k}\right)}\Gamma_k(\alpha+\beta t)^{(a+b)}}{t^{(a+b)\beta}(1-q)^{(ab)\beta(1-t)}\Gamma_q(\alpha+\beta t)^{(ab)}} \le \frac{\Gamma_k(\alpha+\beta t)^{(a+b)}}{\Gamma_q(\alpha+\beta t)^{(ab)}}.$$

Proof. If $t\in[1,\infty)$, then we have $\Upsilon(1)\le\Upsilon(t)$ yielding the result.

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Retracted Article

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